General Physics: Electromagnetism, Correction 6

Exercise 1 :

Find the equivalent capacitance between the points a and b for the group of capacitors connected as shown below. Take $C_1 = 5.00\ \mu$F, $C_2 = 10.00\ \mu$F and $C_3 = 2.00\ \mu$F. What charge is stored on C_3 if the potential difference between the points a and b is 60.0 V?

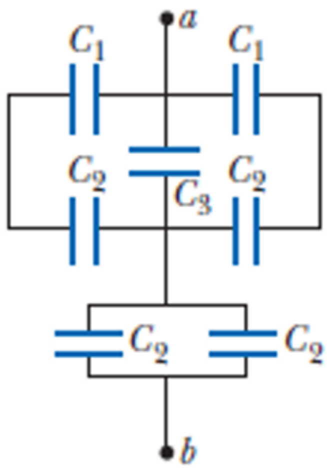

Figure 1: Group of capacitances connected in series and in parallel.

Solution 1 :

In the upper part of the circuit, the C_1 and C_2 capacitors on each side of C_3 are connected in series. The equivalent capacitance C_{12} for these pairs can then be calculated in the following way:

$$C_{12} = \left(\frac{1}{C_1} + \frac{1}{C_2}\right)^{-1} = \frac{C_1 C_2}{C_1 + C_2} = \frac{5.00\mu\text{F} \cdot 10.00\mu\text{F}}{5.00\mu\text{F} + 10.00\mu\text{F}} = 3.33\mu\text{F}. \tag{1}$$

The C_1-C_2 pairs are parallel to the C_3 capacitor, therefore the capacitance of the upper part of the circuit, $C_{\text{eq}}^{\text{up}}$ is found to be:

$$C_{\text{eq}}^{\text{up}} = C_{12} + C_3 + C_{12} = 3.33\mu\text{F} + 2.00\mu\text{F} + 3.33\mu\text{F} = 8.66\mu\text{F}. \tag{2}$$

In the lower part of the circuit, the two C_2 capacitors are in parallel. The equivalent capacity of the lower part of the circuit reads:

$$C_{\text{eq}}^{\text{down}} = C_2 + C_2 = 10.00\mu\text{F} + 10.00\mu\text{F} = 20.00\mu\text{F} \tag{3}$$

Finally, the upper section is in series with the lower section and the total equivalent capacity becomes

$$C_{\text{eq}} = \left(\frac{1}{C_{\text{eq}}^{\text{down}}} + \frac{1}{C_{\text{eq}}^{\text{up}}}\right)^{-1} = \frac{C_{\text{eq}}^{\text{down}} C_{\text{eq}}^{\text{up}}}{C_{\text{eq}}^{\text{down}} + C_{\text{eq}}^{\text{up}}} = \frac{8.66\mu\text{F} \cdot 20\mu\text{F}}{8.66\mu\text{F} + 20.00\mu\text{F}} = 6.05\mu\text{F} \tag{4}$$

Capacitors in series carry the same charge as their equivalent capacitor. Therefore, the upper section (8.66μF) and lower section (20.00μF) carry the same charge as a 6.05μF capacitor:

$$Q_{\text{up}} = Q_{\text{eq}} = C_{\text{eq}}\Delta V = 6.05\mu\text{F} \times 60\text{V} = 363\mu\text{C} \tag{5}$$

The upper section consists of two $C_s(3.33\mu F)$ capacitors and C_3 capacitor that are in parallel. Now the voltage is the same as that across a $8.66\mu F$:

$$\Delta V_3 = \Delta V^{\text{up}} = \frac{Q^{\text{up}}}{C_{\text{eq}}^{\text{up}}} = \frac{363\mu\text{C}}{8.66\mu\text{F}} = 41.9\text{V} \tag{6}$$

The charge stored in C_3 is equal to:

$$Q_3 = C_3\Delta V_3 = 2.00\mu\text{F} \times 41.9\text{V} = 83.8\mu\text{C} \tag{7}$$

Exercise 2 :

Consider the configuration shown in the figure below. Find the equivalent capacitance, assuming that all the capacitors have the same capacitance C.

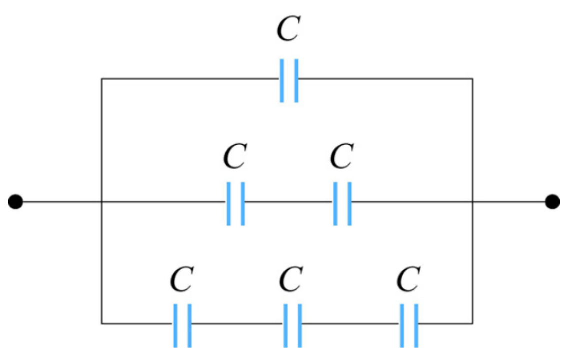

Figure 2: Combination of Capacitors

Solution 2 :

To solve this exercise, one can proceed as in the previous exercise. The circuit is divided in three rows where the capacitors are in series.

The total capacitance of the upper row is simply $C_1 = C$.

The total capacitance of the central row is given by

$$C_2 = \left(\frac{1}{C} + \frac{1}{C}\right)^{-1} = \frac{C}{2}. \tag{8}$$

The total capacitance of the lower row is given by

$$C_3 = \left(\frac{1}{C} + \frac{1}{C} + \frac{1}{C}\right)^{-1} = \frac{C}{3}. \tag{9}$$

Finally, the total capacitance of the circuit is given by the parallel of C_1, C_2, and C_3, that read

$$C_{\mathrm{eq}} = C_1 + C_2 + C_3 = \frac{11}{6}C. \tag{10}$$

Exercise 3 :

Two dielectrics with dielectric constants κ_1 and κ_2 each fill half the space between the plates of a parallel-plate capacitor as shown in the figure below. Each plate has an area A and the plates are separated by a distance d. Compute the capacitance for the system.

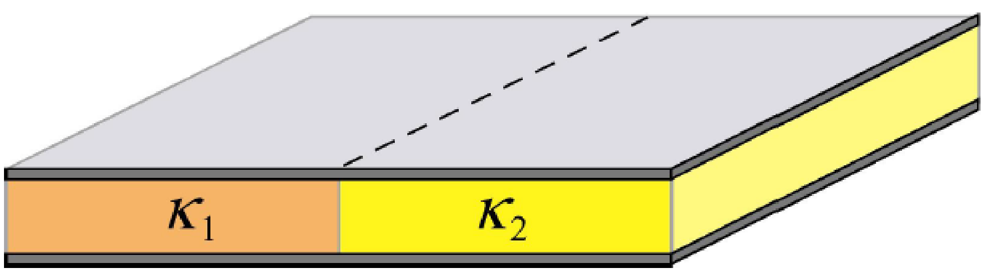

Figure 3: Capacitor filled with two different dielectrics

Solution 3 :

Since the potential difference on each half of the capacitor is the same, we can treat the system as being composed of two capacitors connected in parallel. Thus, the capacitance of the system is

$$C = C_1 + C_2. \tag{11}$$

We can now compute C_j with the help of the formula $C_j = \kappa_j \varepsilon_0 A/2d$, $j = 1, 2$. The capacitance for the system reads

$$C = \frac{\varepsilon_0 A}{2d}(\kappa_1 + \kappa_2). \tag{12}$$

Exercise 4 :

Consider a conducting spherical shell with an inner radius a and outer radius c. Let the space between two surfaces be filled with two different dielectric materials so that the dielectric constant is κ_1 between a and b, and κ_2 between b and c, as shown in the figure below. Determine the capacitance of this system.

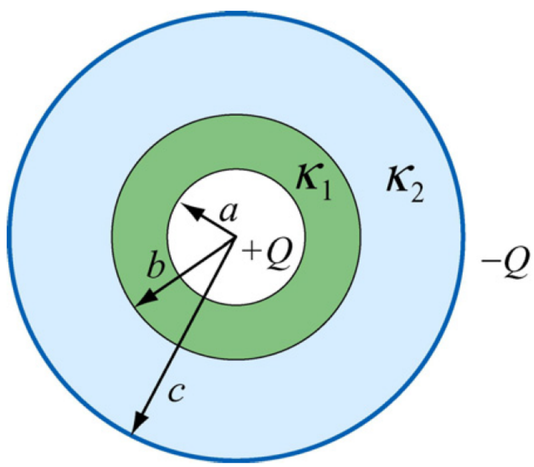

Figure 4: Spherical capacitor filled with dielectrics.

Solution 4 :

The system can be treated as two capacitors connected in series, since the total potential difference across the capacitors is the sum of potential differences across individual capacitors.

First, we derive the capacitance for a spherical capacitor of inner radius r_1 and outer radius r_2 filled with dielectric with dielectric constant κ, and with charges $+Q$ and $-Q$ on the inner and outer sphere. To do this, we use the Gauss law. We have that

$$\int_{\partial \mathcal{V}} \vec{E} \cdot d\vec{\sigma} = \frac{Q}{\kappa \varepsilon_0}, \tag{13}$$

that leads to $4\pi r^2 E(r) = Q/\kappa\varepsilon_0$ if $r_1 < r < r_2$. The total potential difference between r_1 and r_2 reads

$$V = -\int_{r_1}^{r_2} dr E(r) = \frac{Q}{4\pi\kappa\varepsilon_0}\left(\frac{1}{r_1} - \frac{1}{r_2}\right). \tag{14}$$

If we finally apply the definition of capacitance $C = Q/V$ we get

$$C = 4\pi\varepsilon_0\kappa\frac{r_1 r_2}{r_2 - r_1}. \tag{15}$$

The total capacitance then reads

$$C_{\text{eq}} = \left(\frac{1}{C_1} + \frac{1}{C_2}\right)^{-1} = \left(\frac{b-a}{4\pi\varepsilon_0\kappa_1 ab} + \frac{c-b}{4\pi\varepsilon_0\kappa_2 cb}\right)^{-1} = \frac{4\pi\varepsilon_0\kappa_1\kappa_2 abc}{\kappa_2 c(b-a) + \kappa_1 a(c-b)}. \tag{16}$$

Notice that if $\kappa_1, \kappa_2 \to 1$ one gets

$$C \to \frac{4\pi\varepsilon_0 ac}{c-a}, \tag{17}$$

that is the capacitance for a spherical capacitor of inner radius a and outer radius c.

Exercise 5 :

Consider an air-filled parallel-plate capacitor with one plate connected to a spring having a force constant k, and another plate held fixed. The system rests on a table top as shown in the figure below. If the charges placed on plates a and b are $+Q$ and $-Q$, how much does the spring expand?

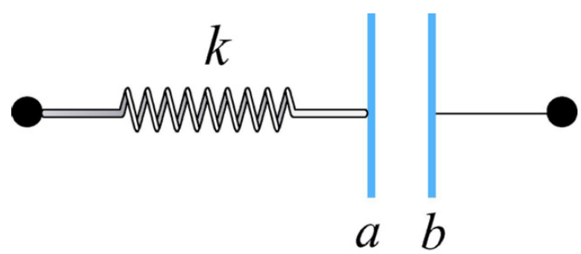

Figure 5: Capacitor connected to a spring.

Solution 5 :

The spring force acting on plate a is given by the Hooke's law, $F_s = -kx$.

At the same time, the electrostatic force due to the electric field within the capacitor reads $F_{\text{el}} = QE = Q\sigma/(2\varepsilon_0) = Q^2/(2\varepsilon_0 A)$, where $\sigma = Q/A$ and A is the capacitor's area.

Since the system is at rest, the Newton's second law imposes $F_s + F_{\text{el}} = 0$, that gives, $x = Q^2/(2kA\varepsilon_0)$.

Exercise 6 :

We consider a parallel-plate capacitor with plate separation d and plate area A.

1. Find the capacitance of the device when an uncharged metallic slab of thickness a is inserted midway between the plates. What is the capacitance in the limit where $a \to 0$?

2. Find the capacitance of the device when a slab of dielectric material of dielectric constant k and thickness fd is inserted between the plates, where f is a fraction between 0 and 1. Express the solution in terms of the capacitance C_0 in the absence of the dielectric ($C_0 = \epsilon_0 A/d$).

Solution 6 :

1. The charge present on one plate of the capacitor induces a charge of equal magnitude and opposite sign on the near side of the slab. Consequently, the net charge on the slab remains zero and the electric field inside the slab is zero. The planes of charge on the metallic slab's upper and lower edges are identical to the distribution of charges on the plate of the capacitor. Therefore, we can model the edges of the slab as conducting planes and the bulk of the slab as a wire. As a result, the device is equivalent to two capacitors in series each having a plate separation $(d-a)/2$.

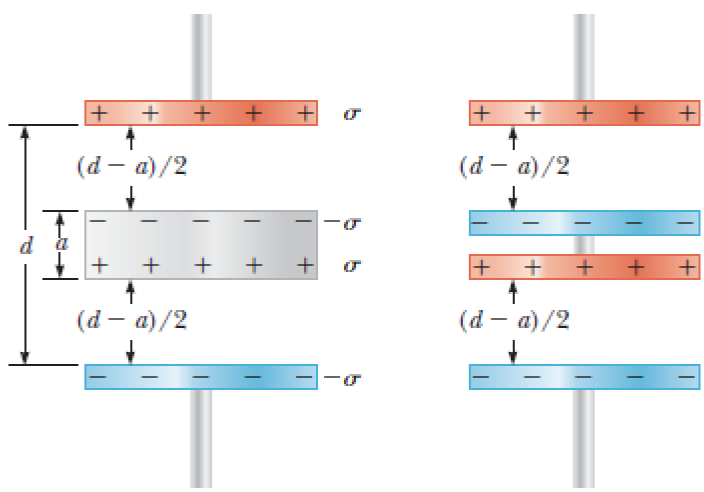

Figure 6: Left panel: A parallel-plate capacitor of plate separation d partially filled with a metallic slab of thickness a. Right panel: The system is equivalent to two capacitors in series each having a plate separation $(d-a)/2$

$$\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2} = \frac{1}{\frac{\epsilon_0 A}{(d-a)/2}} + \frac{1}{\frac{\epsilon_0 A}{(d-a)/2}}. \tag{18}$$

Solving for C gives the total equivalent capacitance,

$$C = \frac{\epsilon_0 A}{d-a}. \tag{19}$$

In the limit where $a \to 0$

$$C = \lim_{a\to 0}\left(\frac{\epsilon_0 A}{d-a}\right) = \frac{\epsilon_0 A}{d}. \tag{20}$$

This result indicates that we can insert an infinitesimally thin metallic sheet between the plates of a capacitor without affecting the capacitance.

2. Similarly to the previous question, the device can be modeled as series combination of two capacitors. One capacitor has plate separation fd and is filled with dielectric; the other one has plate separation $(1-f)d$ and has hair between its plate. The total capacitance

$$\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2} = \frac{1}{\frac{\kappa\epsilon_0 A}{fd}} + \frac{1}{\frac{\epsilon_0 A}{(1-f)d}} \tag{21}$$

We have

$$\frac{1}{C} = \frac{fd}{\kappa\epsilon_0 A} + \frac{\kappa(1-f)d}{\kappa\epsilon_0 A} = \frac{f+\kappa(1-f)}{\kappa}\frac{d}{\epsilon_0 A}, \tag{22}$$

Solving for C gives the total equivalent capacitance

$$C = \frac{\kappa}{f+\kappa(1-f)}\frac{\epsilon_0 A}{d} = \frac{\kappa}{f+\kappa(1-f)}C_0. \tag{23}$$

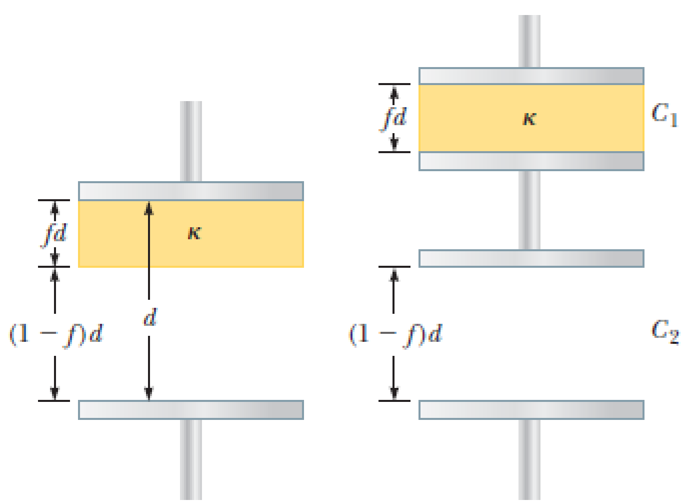

Figure 7: Left panel: A parallel-plate capactior of plate separation d filled with a dielectric of thickness fd. Right panel: The system is equivalent to two capacitors in series, one with separation fd, the other one with separation $(1-f)d$.